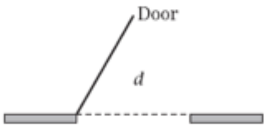
Section 2.1

Grade: «grade»

Subject: «subject»

Date: «date»

Exploration 1: Instantaneous Rate of Change of a Function



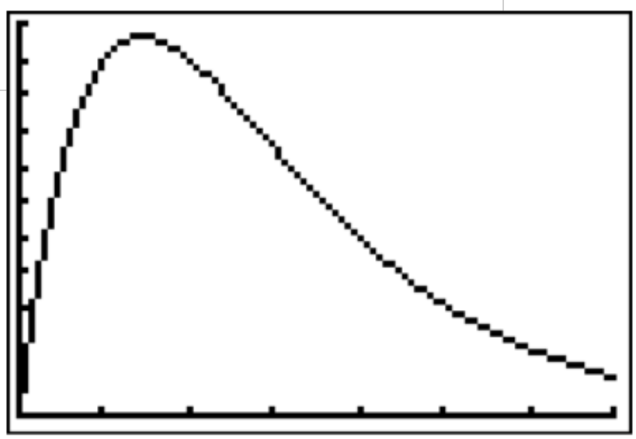
The diagram shows a door with an automatic closer. At time $t = 0$ s, someone pushes the door. It swings open, slows down, stops, starts closing, then slams shut at time $t = 7$ s. As the door is in motion, the number of degrees, d, it is from its closed position depends on t.

1. Sketch a reasonable graph of d versus t.

2. Suppose that d is given by the equation

$$d = 200t \cdot 2^{-t}$$

Plot this graph on your grapher. Sketch the results here.



3. Make a table of values of d for each second from $t = 0$ through $t = 10$. Round to the nearest 0.1°.

t	d
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

X	Y1	
0	0	
1	100	
2	100	
3	75	
4	50	
5	31.25	
6	18.75	

Press + for ΔTbl

X	Y1	
4	50	
5	31.25	
6	18.75	
7	10.938	
8	6.25	
9	3.5156	
10	1.9531	

X=10

4. At time $t = 1$ s, does the door appear to be opening or closing? How do you tell?

5. What is the average rate at which the door is moving for the time interval [1, 1.1]? Based on your answer, does the door seem to be opening or closing at time $t = 1$? Explain.

6. By finding average rates using the time intervals [1, 1.01], [1, 1.001], and so on, make a conjecture about the *instantaneous* rate at which the door is moving at time $t = 1$ s.

```
(Y₁(1.01)-Y₁(1)▸
      30.23420391
(Y₁(1.001)-Y₁(1▸
      30.64000835
```

7. In calculus you will learn by four methods:
   - algebraically,
   - numerically,
   - graphically,
   - verbally (talking and writing).

   What did you learn as a result of doing this Exploration that you did not know before?

Meaning of Derivative

The derivative of function $f(x)$ at $x = c$ is the *instantaneous rate of change* of $f(x)$ with respect to x at $x = c$. It is found

- Numerically, by taking the *limit* of the average rate over the interval from c to x as x approaches c
- Graphically, by finding the slope of the line tangent to the graph at $x = c$

Average and Instantaneous Speed

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the elapsed time. The unit of measure is length per unit time—kilometers per hour, feet per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

$y = 16t^2$ y is distance in ft and t is time in seconds

1 Answer?

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EXAMPLE 2 **Finding an Instantaneous Speed**

Find the speed of the rock in Example 1 at the instant $t = 2$.

Solve Numerically We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}. \tag{1}$$

Table 2.1 Average Speeds over Short Time Intervals Starting at $t = 2$

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

Length of Time Interval, h (sec)	Average Speed for Interval $\Delta y/\Delta t$ (ft/sec)
1	80
0.1	65.6
0.01	64.16
0.001	64.016
0.0001	64.0016
0.00001	64.00016

Confirm Algebraically If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h.$$

Exploration 2: Graphs of Familiar Functions

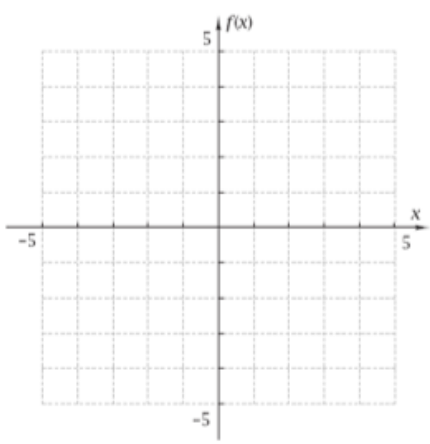
For each function:

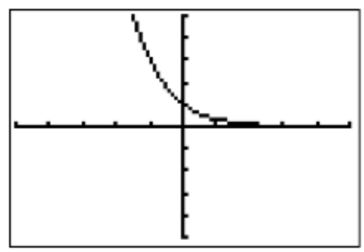
a. Without using your grapher, sketch the graph on the axes provided.

b. Confirm by grapher that your sketch is correct.

c. Tell whether the function is increasing, decreasing, or not changing when $x = 1$. If it is increasing or decreasing, tell whether the rate of change is slow or fast.

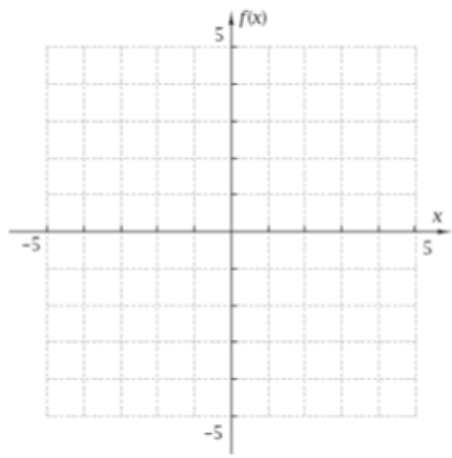
1. $f(x) = 3^{-x}$

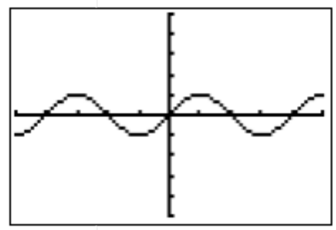




c. decreasing at a slow rate at x = 1

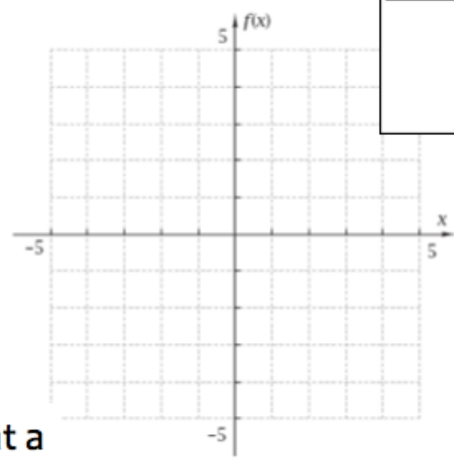
2. $f(x) = \sin \frac{\pi}{2}x$

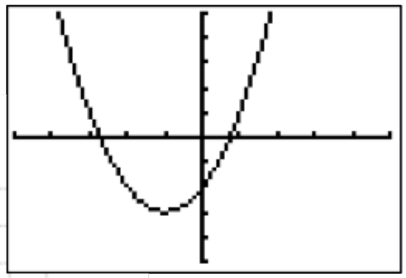




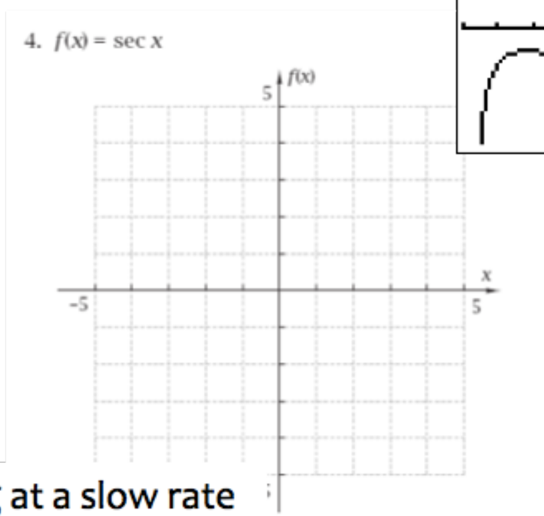
Not Changing

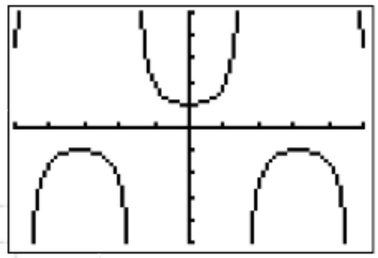
3. $f(x) = x^2 + 2x - 2$



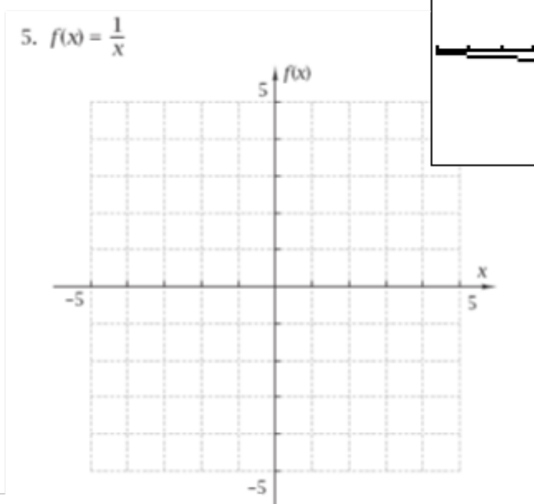


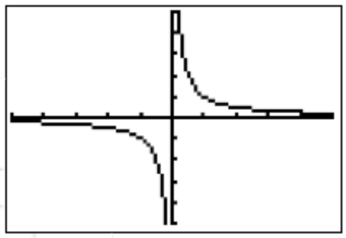
c. Increasing at a fast rate at x = 1





c. Increasing at a slow rate
at x=1





c. Decreasing at a slow rate at x=1

$$\lim_{x \to 2} \frac{x^4 - 1}{x - 1}$$

$= 15$

$f(x) = \frac{x^4 - 1}{x - 1}$

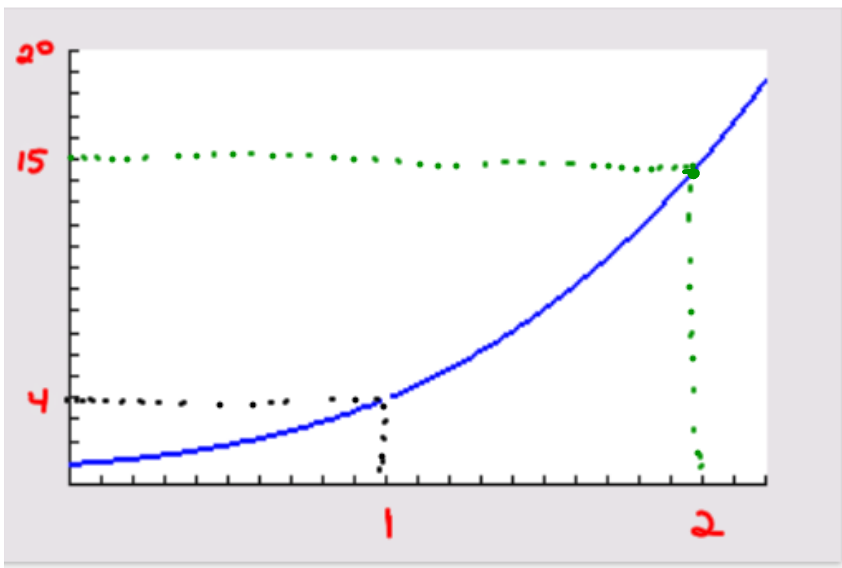
$f(2) = 15$ value at $x = 2$

$f(1) = DNE$

Introduction to Limits of Functions Lab

$\lim_{x \to 1} f(x)$

$= 4$



$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1}$$

$\downarrow$

$$\frac{(x^2-1)(x^2+1)}{x-1}$$

$$\frac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{x-1}}$$

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{x-1}}$$

$$(1+1)(1^2+1) = 4$$

Verbal Definition of Limit

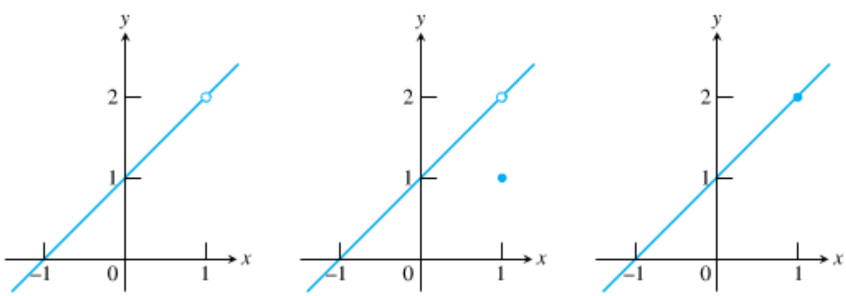
L is the limit of $f(x)$ as x approaches c

if and only if

L is the *one* number you can keep $f(x)$ arbitrarily close to

just by keeping x close enough to c, but not equal to c.

Figure 2.2 illustrates the fact that the existence of a limit as $x \to c$ never depends on how the function may or may not be defined at c. The function f has limit 2 as $x \to 1$ even though f is not defined at 1. The function g has limit 2 as $x \to 1$ even though $g(1) \neq 2$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

Figure 2.2 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$

THEOREM 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $$\lim_{x \to c} (f(x) + g(x)) = L + M$$
   The limit of the sum of two functions is the sum of their limits.
2. *Difference Rule:* $$\lim_{x \to c} (f(x) - g(x)) = L - M$$
   The limit of the difference of two functions is the difference of their limits.
3. *Product Rule:* $$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$
   The limit of a product of two functions is the product of their limits.
4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$
   The limit of a constant times a function is the constant times the limit of the function.
5. *Quotient Rule:* $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \ M \neq 0$$
   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. *Power Rule:* If r and s are integers, $s \neq 0$, then
   $$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$
   provided that $L^{r/s}$ is a real number.

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

EXAMPLE 3 **Using Properties of Limits**

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$, and the properties of limits to find the following limits.

(a) $\lim_{x\to c} (x^3 + 4x^2 - 3)$

$= c^3 + 4c^2 - 3$

(b) $\lim_{x\to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

$= \dfrac{c^4 + c^2 - 1}{c^2 + 5}$

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THEOREM 2 Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

EXAMPLE 4 **Using Theorem 2**

2 Answer?

(a) $\lim_{x \to 3} [x^2(2-x)]$ $\quad$ $3^2(2-3)$

$= -9$

EXAMPLE 4 **Using Theorem 2**

3 Answer?

(b) $\lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2}$

$$\frac{(2)^2 + 2(2) + 4}{2 + 2}$$

$= 3$

4 Answer?

EXAMPLE 5 Using the Product Rule

Determine $\lim_{x \to 0} \frac{\tan x}{x}$.

hint: $\tan x = \frac{\sin x}{\cos x}$

$$\lim_{x \to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 1 \cdot 1$$

$$= 1$$

EXAMPLE 6 Exploring a Nonexistent Limit

Use a graph to show that

$$\lim_{x \to 2} \frac{x^3 - 1}{x - 2}$$

does not exist.

Use scale [–10, 10] by [–100, 100]

not approaching same

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right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

EXAMPLE 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int}\, x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 2.5. For example,

$$\lim_{x \to 3^+} \text{int}\, x = 3 \quad \text{and} \quad \lim_{x \to 3^-} \text{int}\, x = 2.$$

The limit of int x as x approaches an integer n from the right is n, while the limit as x approaches n from the left is $n - 1$.

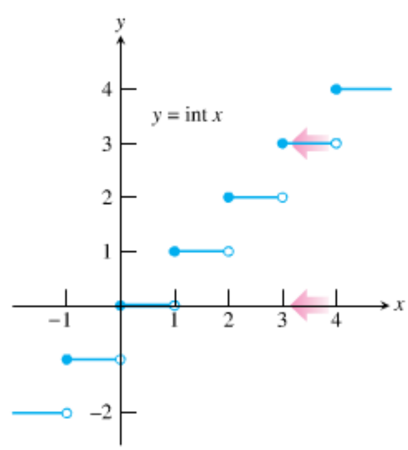

Figure 2.5 At each integer, the greatest integer function $y = \text{int}\, x$ has different right-hand and left-hand limits. (Example 7)

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THEOREM 3 One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c^-} f(x) = L.$$

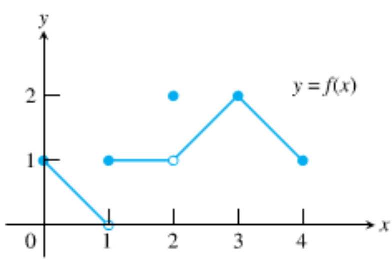


Figure 2.6 The graph of the function

$$f(x) = \begin{cases} -x+1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x-1, & 2 < x \le 3 \\ -x+5, & 3 < x \le 4. \end{cases}$$

(Example 8)

EXAMPLE 8 Exploring Right- and Left-Hand Limits

All the following statements about the functi[redacted]ed in Figure 2.6 are true.

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$. [redacted]

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$, [redacted]

At $x = 2$: [redacted]

At $x = 3$: [redacted]

At $x = 4$: [redacted]

At noninteger values of c between 0 and 4[redacted]

THEOREM 4 The Sandwich Theorem

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

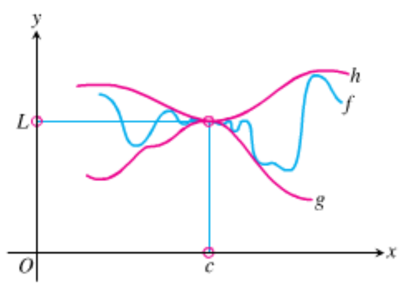
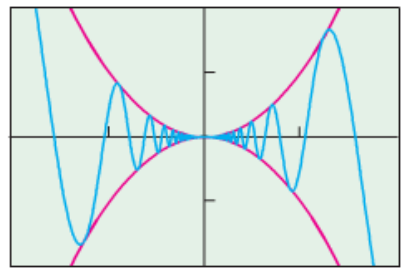


Figure 2.7 Sandwiching f between g and h forces the limiting value of f to be between the limiting values of g and h.

EXAMPLE 9 Using the Sandwich Theorem

Show that $\lim_{x \to 0} [x^2 \sin(1/x)] = 0$.



[−0.2, 0.2] by [−0.02, 0.02]

Figure 2.8 The graphs of $y_1 = x^2$, $y_2 = x^2 \sin(1/x)$, and $y_3 = -x^2$. Notice that $y_3 \le y_2 \le y_1$. (Example 9)